7.5 Glide Reflections and Compositions

What you should learn

GOAL 1 Identify glide reflections in a plane.

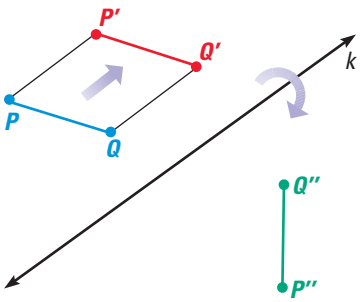
GOAL 2 Represent transformations as compositions of simpler transformations.

Why you should learn it

▼ Compositions of transformations can help when creating patterns in **real life**, such as the decorative pattern below and in **Exs. 35–37**.

GOAL 1 USING GLIDE REFLECTIONS

A translation, or glide, and a reflection can be performed one after the other to produce a transformation known as a *glide reflection*. A **glide reflection** is a transformation in which every point P is mapped onto a point P'' by the following steps:



1. A translation maps P onto P'.
2. A reflection in a line k parallel to the direction of the translation maps P' onto P''.

As long as the line of reflection is parallel to the direction of the translation, it does not matter whether you glide first and then reflect, or reflect first and then glide.

EXAMPLE 1 *Finding the Image of a Glide Reflection*

Use the information below to sketch the image of $\triangle ABC$ after a glide reflection.

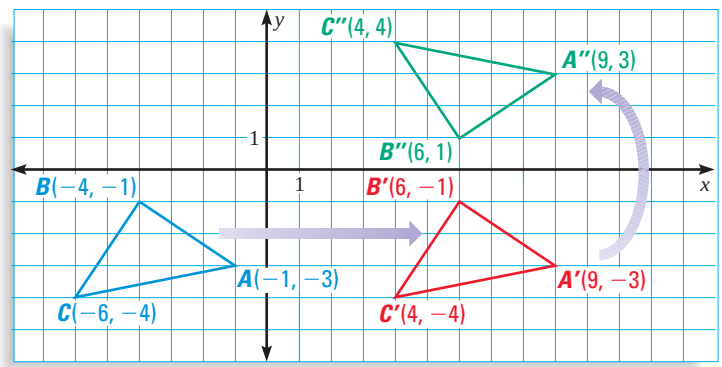
$A(-1, -3)$, $B(-4, -1)$, $C(-6, -4)$

Translation: $(x, y) \rightarrow (x + 10, y)$

Reflection: in the x-axis

SOLUTION

Begin by graphing $\triangle ABC$. Then, shift the triangle 10 units to the right to produce $\triangle A'B'C'$. Finally, reflect the triangle in the x-axis to produce $\triangle A''B''C''$.



In Example 1, try reversing the order of the transformations. Notice that the resulting image will have the same coordinates as $\triangle A''B''C''$ above. This is true because the line of reflection is parallel to the direction of the translation.

GOAL 2 USING COMPOSITIONS

When two or more transformations are combined to produce a single transformation, the result is called a **composition** of the transformations.

THEOREM

THEOREM 7.6 ***Composition Theorem***

The composition of two (or more) isometries is an isometry.

Because a glide reflection is a composition of a translation and a reflection, this theorem implies that glide reflections are isometries. In a glide reflection, the order in which the transformations are performed does not affect the final image. For other compositions of transformations, the order may affect the final image.

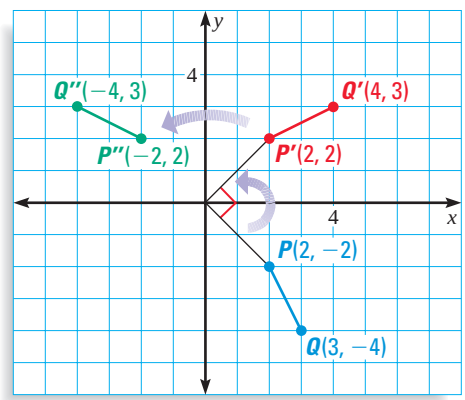
EXAMPLE 2 *Finding the Image of a Composition*

Sketch the image of $\overline{PQ}$ after a composition of the given rotation and reflection.

$P(2, -2)$, $Q(3, -4)$

Rotation: 90° counterclockwise about the origin

Reflection: in the y-axis



SOLUTION

Begin by graphing $\overline{PQ}$. Then rotate the segment 90° counterclockwise about the origin to produce $\overline{P'Q'}$. Finally, reflect the segment in the y-axis to produce $\overline{P''Q''}$.

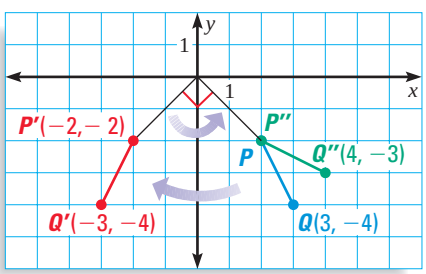
EXAMPLE 3 *Comparing Orders of Compositions*

Repeat Example 2, but switch the order of the composition by performing the reflection first and the rotation second. What do you notice?

SOLUTION

Graph $\overline{PQ}$. Then reflect the segment in the y-axis to obtain $\overline{P'Q'}$. Rotate $\overline{P'Q'}$ 90° counterclockwise about the origin to obtain $\overline{P''Q''}$. Instead of being in Quadrant II, as in Example 2, the image is in Quadrant IV.

▶ The order which the transformations are performed affects the final image.



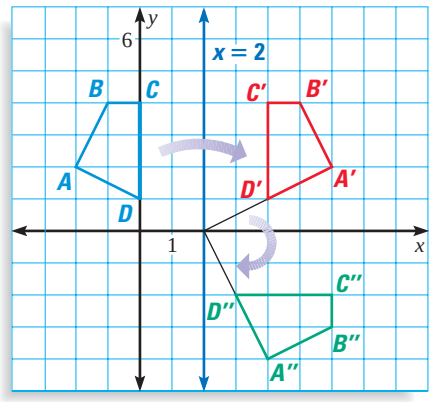
STUDENT HELP

Study Tip

Unlike the addition or multiplication of real numbers, the composition of transformations is not generally commutative.

EXAMPLE 4 Describing a Composition

Describe the composition of transformations in the diagram.



SOLUTION

Two transformations are shown. First, figure $ABCD$ is reflected in the line $x = 2$ to produce figure $A'B'C'D'$. Then, figure $A'B'C'D'$ is rotated 90° clockwise about the point (2, 0) to produce figure $A''B''C''D''$.

EXAMPLE 5 Describing a Composition

PUZZLES The mathematical game pentominoes is a tiling game that uses twelve different types of tiles, each composed of five squares. The tiles are referred to by the letters they resemble. The object of the game is to pick up and arrange the tiles to create a given shape. Use compositions of transformations to describe how the tiles below will complete the 6 × 5 rectangle.

STUDENT HELP

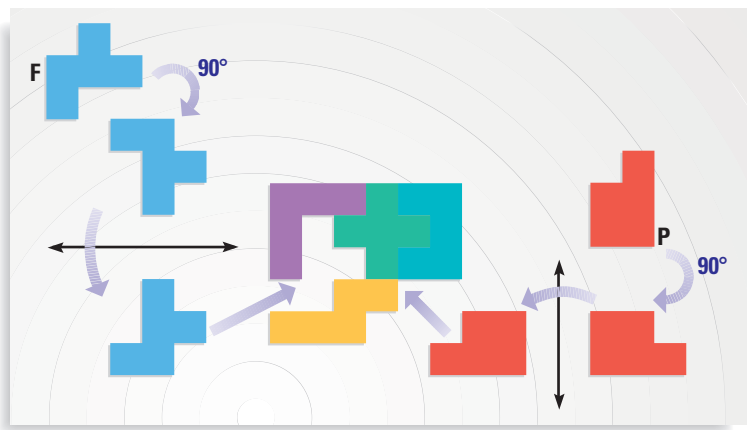
Study Tip

You can make your own pentomino tiles by cutting the shapes out of graph paper.

SOLUTION

To complete part of the rectangle, rotate the F tile 90° clockwise, reflect the tile over a horizontal line, and translate it into place.

To complete the rest of the rectangle, rotate the P tile 90° clockwise, reflect the tile over a vertical line, and translate it into place.



GUIDED PRACTICE

Vocabulary Check ✓

1. In a glide reflection, the direction of the __?__ must be parallel to the line of __?__.

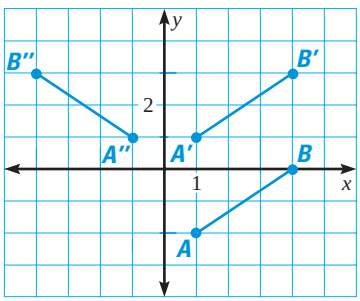
Concept Check ✓

Complete the statement with *always*, *sometimes*, or *never*.

2. The order in which two transformations are performed __?__ affects the resulting image.
3. In a glide reflection, the order in which the two transformations are performed __?__ matters.
4. A composition of isometries is __?__ an isometry.

Skill Check ✓

In the diagram, $\overline{AB}$ is the preimage of a glide reflection.



5. Which segment is a translation of $\overline{AB}$?
6. Which segment is a reflection of $\overline{A'B'}$?
7. Name the line of reflection.
8. Use coordinate notation to describe the translation.

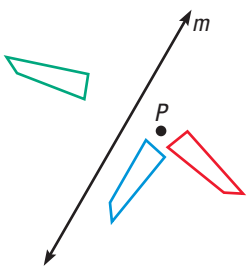
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 816.

LOGICAL REASONING **Match the composition with the diagram, in which the blue figure is the preimage of the red figure and the red figure is the preimage of the green figure.**

A.



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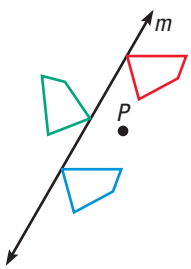
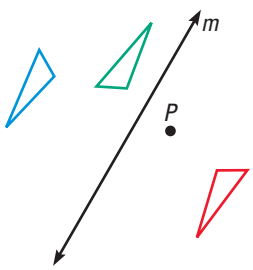


C.



9. Rotate about point P, then reflect in line m.
10. Reflect in line m, then rotate about point P.
11. Translate parallel to line m, then rotate about point P.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 9–15
Example 2: Exs. 16–19
Example 3: Exs. 20, 21
Example 4: Exs. 22–25
Example 5: Ex. 38

FINDING AN IMAGE **Sketch the image of $A(-3, 5)$ after the described glide reflection.**

12. **Translation:** $(x, y) \rightarrow (x, y - 4)$
Reflection: in the y-axis
13. **Translation:** $(x, y) \rightarrow (x + 4, y + 1)$
Reflection: in $y = -2$
14. **Translation:** $(x, y) \rightarrow (x - 6, y - 1)$
Reflection: in $x = -1$
15. **Translation:** $(x, y) \rightarrow (x - 3, y - 3)$
Reflection: in $y = x$

STUDENT HELP

INTERNET HOMEWORK HELP

Visit our Web site www.mcdougallittell.com for help with Exs. 16–19.

SKETCHING COMPOSITIONS **Sketch the image of $\triangle PQR$ after a composition using the given transformations in the order they appear.**

16. $P(4, 2)$, $Q(7, 0)$, $R(9, 3)$
Translation: $(x, y) \rightarrow (x - 2, y + 3)$
Rotation: 90° clockwise about $T(0, 3)$

17. $P(4, 5)$, $Q(7, 1)$, $R(8, 8)$
Translation: $(x, y) \rightarrow (x, y - 7)$
Reflection: in the y-axis

18. $P(-9, -2)$, $Q(-9, -5)$, $R(-5, -4)$
Translation: $(x, y) \rightarrow (x + 14, y + 1)$
Translation: $(x, y) \rightarrow (x - 3, y + 8)$

19. $P(-7, 2)$, $Q(-6, 7)$, $R(-2, -1)$
Reflection: in the x-axis
Rotation: 90° clockwise about origin

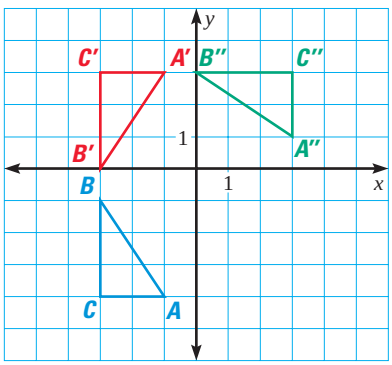
REVERSING ORDERS **Sketch the image of $\overline{FG}$ after a composition using the given transformations in the order they appear. Then, perform the transformations in reverse order. Does the order affect the final image?**

20. $F(4, -4)$, $G(1, -2)$
Rotation: 90° clockwise about origin
Reflection: in the y-axis

21. $F(-1, -3)$, $G(-4, -2)$
Reflection: in the line $x = 1$
Translation: $(x, y) \rightarrow (x + 2, y + 10)$

DESCRIBING COMPOSITIONS **In Exercises 22–25, describe the composition of the transformations.**

22.



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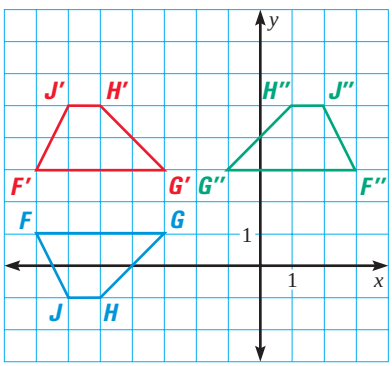

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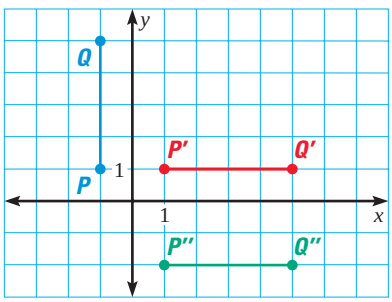
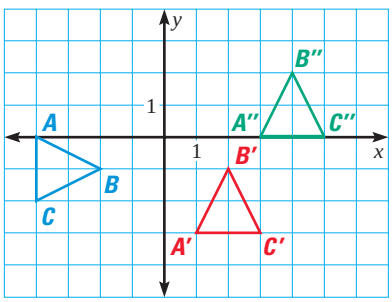


25.



26. *Writing* Explain why a glide reflection is an isometry.

27. **LOGICAL REASONING** Which are preserved by a glide reflection?

A. distance **B.** angle measure **C.** parallel lines

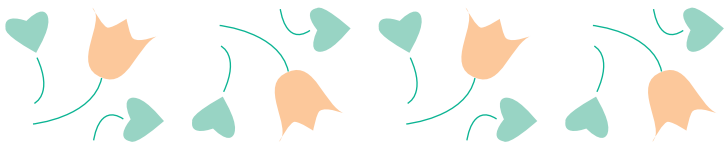
28. **TECHNOLOGY** Use geometry software to draw a polygon. Show that if you reflect the polygon and then translate it in a direction that is *not* parallel to the line of reflection, then the final image is *different* from the final image if you perform the translation first and the reflection second.

CRITICAL THINKING **In Exercises 29 and 30, the first translation maps J to J' and the second maps J' to J''. Find the translation that maps J to J''.**

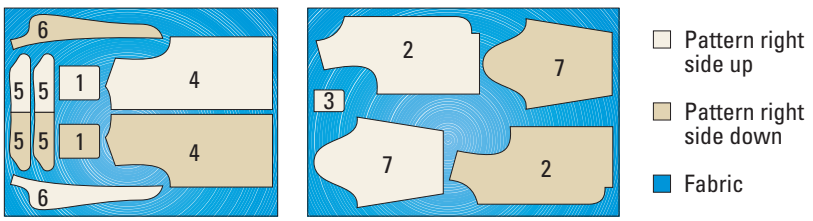
29. **Translation 1:** $(x, y) \rightarrow (x + 7, y - 2)$
Translation 2: $(x, y) \rightarrow (x - 1, y + 3)$
Translation: $(x, y) \rightarrow (\_?\_, \_?\_)$

30. **Translation 1:** $(x, y) \rightarrow (x + 9, y + 4)$
Translation 2: $(x, y) \rightarrow (x + 6, y - 4)$
Translation: $(x, y) \rightarrow (\_?\_, \_?\_)$

31. **STENCILING A BORDER** The border pattern below was made with a stencil. Describe how the border was created using one stencil four times.



CLOTHING PATTERNS **The diagram shows the pattern pieces for a jacket arranged on some blue fabric.**



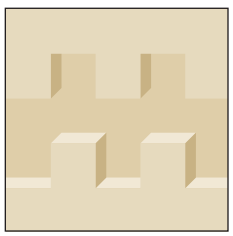
32. Which pattern pieces are translated?

33. Which pattern pieces are reflected?

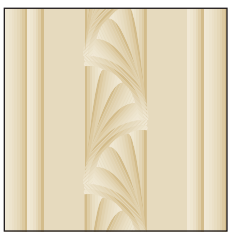
34. Which pattern pieces are glide reflected?

ARCHITECTURE **In Exercises 35–37, describe the transformations that are combined to create the pattern in the architectural element.**

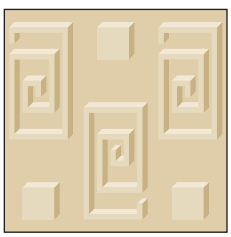
35.



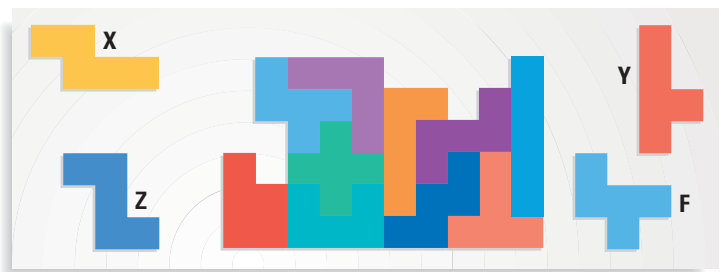
36.



37.



38. **PENTOMINOES** Use compositions of transformations to describe how to pick up and arrange the tiles to complete the 6×10 rectangle.



FOCUS ON CAREERS

ARCHITECTURAL HISTORIANS play an important role in the renovation of an old building. Their expertise is used to help restore a building to its original splendor.

CAREER LINK
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Test Preparation

39. MULTI-STEP PROBLEM Follow the steps below.

a. On a coordinate plane, draw a point and its image after a glide reflection that uses the x-axis as the line of reflection.

b. Connect the point and its image. Make a conjecture about the midpoint of the segment.

c. Use the coordinates from part (a) to prove your conjecture.

d. CRITICAL THINKING Can you extend your conjecture to include glide reflections that do not use the x-axis as the line of reflection?

Challenge

40. USING ALGEBRA Solve for the variables in the glide reflection of $\triangle JKL$ described below.

$J(-2, -1)$
$K(-4, 2a)$
$L(b - 6, 6)$

Translate $(x, y) \rightarrow (x + 3, y)$

$J'(c + 1, -1)$
$K'(5d - 11, 4)$
$L'(2, 4e)$

Reflect in x-axis

$J''(1, -f)$
$K''(-1, 3g + 5)$
$L''(h + 4, -6)$

EXTRA CHALLENGE
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MIXED REVIEW

ANALYZING PATTERNS Sketch the next figure in the pattern. (Review 1.1 for 7.6)

41.

42.

43.

44.

COORDINATE GEOMETRY In Exercises 45–47, decide whether $\square PQRS$ is a *rhombus*, a *rectangle*, or a *square*. Explain your reasoning. (Review 6.4)

45. $P(1, -2)$, $Q(5, -1)$, $R(6, -5)$, $S(2, -6)$

46. $P(10, 7)$, $Q(15, 7)$, $R(15, 1)$, $S(10, 1)$

47. $P(8, -4)$, $Q(10, -7)$, $R(8, -10)$, $S(6, -7)$

48. ROTATIONS A segment has endpoints $(3, -8)$ and $(7, -1)$. If the segment is rotated 90° counterclockwise about the origin, what are the endpoints of its image? (Review 7.3)

STUDYING TRANSLATIONS Sketch $\triangle ABC$ with vertices $A(-9, 7)$, $B(-9, 1)$, and $C(-5, 6)$. Then translate the triangle by the given vector and name the vertices of the image. (Review 7.4)

49. $\langle 3, 2 \rangle$

50. $\langle -1, -5 \rangle$

51. $\langle 6, 0 \rangle$

52. $\langle -4, -4 \rangle$

53. $\langle 0, 2.5 \rangle$

54. $\langle 1.5, -4.5 \rangle$